ON THE STABILITY OF SOLUTIONS OF DIFFERENTIAL EQUATIONS OF SECOND ORDER

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PMM Vol.23, No.3, 1959, pp. 596-598

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(Received 23 February 1959)

In the present paper several theorems are proposed on the stability in the sense of Liapunov [1] of the solutions of differential equations of second order. As a source for this paper the works [2-5] can be given.

Let us introduce first certain concepts. Let the functions $v_1(s, y)$, ..., $V_n(x, y)$ be defined, continuous and single-values in the regions G_1, ..., G_n, respectively. The mutual disposition of these regions is such that any two consecutive regions G_1, ..., G_n, G_1 contain a certain unique Jordan curve passing through the origin of the coordinates (such curves will be called L-curves). With respect to the functions $V_k(x, y)$ $(k = 1, ..., n)$ it is further assumed that they are positive in the regions of their definition, except at the origin of coordinates where they vanish, and that they vary strictly monotonically along the bounding curves.

Under these assumptions there exists a neighborhood of the origin of the coordinates such that, starting from an arbitrary point A situated in this neighborhood and on the L-curve, it is possible to construct, selecting a definite sense for going around the origin of the coordinates, a connected curve consisting of pieces along which one of the considered functions assumes a constant value. The curve constructed in this way will intersect again the L-curve at a certain point B which, in general, does not coincide with point A. If we assume that in the neighborhood under consideration such a coincidence is impossible and if we denote by $V(A)$ $V(B)$ the values of one of these functions defined on the initial L-curve at points A and B, respectively, then one of the inequalities $V(A) < V(B)$ or $V(A) > V(B)$ will hold. Let it be the former, for instance, satisfied. Due to the assumptions made with respect to the functions $V_k(x, y)$ $(k = 1, ..., n)$ the sense of the obtained inequality does not change, no matter on what L-curve the point A is taken, provided only that in constructing the curves we keep the same sense for going around the origin of the coordinates.

Definition. Choosing a definite direction we shall say that the set of function $V_k(x, y)$ $(k - 1, \ldots, n)$ possesses positive or negative rotation depending on whether the first or the second inequality holds.

Consider the system of differential equations

$$\frac{dx}{dt} = X(x, y), \qquad \frac{dy}{dt} = Y(x, y) \tag{1}$$

It is assumed that the functions $X(x, y)$ and $Y(x, y)$ are continuous in a certain neighborhood of the origin of coordinates and vanish when $x = y = 0$.

Theorem 1. If it is possible to find for the system (1) two sets of functions, possessing opposite rotations and being such that the time derivative of each of these functions by virtue of (1) is negative or vanishes identically in the region of its definition, then the zero solution of the system (1) is stable.

Proof. Without loss of generality of the argument we can assume that one of the L-curves of the sets of functions is common to both sets. Starting from a point A on such an L-curve, on the basis of the functions which make up the set of functions with positive rotation, construct the curve which was mentioned above. This curve will again intersect the L-curve at a certain point B.

Starting from point B, on the basis of the functions which make up the set with negative rotation, construct an analogous curve. The last curve will necessarily intersect the first one before it intersects the initial L-curve for the second time. This is so because the sets of functions possess opposite rotations.

As a result of this construction we obtain a closed curve, consisting of pieces on which one of the considered functions assumes a constant value. Since the diameter of a similar curve can be made arbitrarily small (let us note for the general case that this and only this fact is used in the proof), then, in order to prove the theorem, it is sufficient to show that the integral curves corresponding to initial points from the regions bounded by such closed curves cannot leave these regions. This, however, follows from the fact that any point (x_0, y_0) of such a closed curve can be enclosed in so small a circular neighborhood that for all functions $V(x, y)$, contained in the sets and determined in this neighborhood, by virtue of the identity

$$V - V_0 - \int_{t_0}^{t} \frac{dV}{dt}\, dt, \qquad V_0 = V(x_0, y_0) \tag{2}$$

and the conditions of the theorem, we have $V \leqslant V_0$, so long as the integral curve passing through the point (x_0, y_0) remains in the above mentioned neighborhood.

Remark. It is not difficult to see that a singular point of the type of a center cannot be detected by means of Theorem 1.

Theorem 2. If it is possible to find for the system (1) two sets of functions, possessing opposite rotations and being such that the time derivative of each of these functions by virtue of (1) is negative definite in the region of its definition, then the zero solution of system (1) is asymptotically stable.

Proof. On the basis of Theorem 1 the above mentioned solution is stable. Assume that this solution is not asymptotically stable. Then, in any arbitrarily small neighborhood of the origin of the coordinates there exists an integral curve which lies completely in a ring-shaped region, not containing the origin of coordinates. Such an integral curve cannot remain indefinitely in one of the regions of definition of the functions contained in the sets (2).

Consequently, this integral curve intersects some one of the L-curves an infinite number of times. From this set of points of intersection select a convergent sequence $\{P_k\}$. The limit of this sequence, by virtue of the assumptions on the behavior of the integral curve, is different from the origin of coordinates. Denote this limit point by P. Consider a circle with the center at point P and so small a radius δ that inside this circle are defined all the functions V of the considered sets which are defined at point P.

Since these functions are continuous, then along the sequence of points $\{P_k\}$ every one of them must have a definite limit. Denote by V_k the values of the functions at the point P_k and by V_δ the values of the functions at the point through which the integral curve leaves the above circle at the instant immediately following the instant of intersection of the integral curve and the L-curve at the point P_k. Then for sufficiently large values of k by virtue of the assumptions of the theorem we shall have

$$|V_\delta - V_k| \geqslant \frac{\delta' m}{M}$$

Here δ' denotes some positive number, smaller than δ and independent of k; m is the smallest of the least values of the functions $|dV/dt|$ and M the largest value of $(x^2 + y^2)^{1/2}$ in the same circle (X, Y are the right-hand sides of system (1)).

Further, from the condition that the time derivatives of the functions making up the sets, by virtue of system (1), are functions of definite signs in their regions of definition, we conclude that through every point situated in a sufficiently small neighborhood of the origin of the coordinates, passes a closed curve mentioned in the proof of Theorem 1.

From this and on the basis of the obtained inequality we conclude that at least for two functions, defined in the indicated neighborhood of point P, the inequality

$$|V_{k+1} - V_k| \geqslant h$$

holds, where h is a certain positive constant which does not depend on k. This, however, contradicts the fact that all numerical sequences $\{V_k\}$ must possess definite limits. Having in mind the rings which can be made from the above mentioned closed curves, we arrive at the proof of the theorem.

Theorem 3. If it is possible to find for system (1) two sets of functions, possessing opposite rotations and being such that the time derivative of each function by virtue of system (1) is a positive definite function in the region of its definition, then the zero solution of system (1) is unstable.

The proof is analogous to the proof of Theorem 2.

Remark. Theorem 1 remains valid also for the case when the right-hand sides of system (1) depend on time.

Theorems 2 and 3 in this case remain valid provided the right-hand sides of system (1) are bounded and we understand by a function of definite sign in its region of definition a function of definite sign in the sense of Chetaev [3].

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Translated by E.L.